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On the holonomy group associated with analytic continuations of solutions for integrable systems

Hidekazu Ito

—Dedicated to Professor Kenichi Shiraiwa

Abstract. Necessary conditions for complex Hamiltonian systems to be integrable are considered in connection with holonomy representations of the Riemann surfaces of solutions. They are concerned with analytic continuations of solutions near those satisfying some non-resonance condition. We prove that if the system is integrable, there exists a system of local coordinates in which all Poincaré maps associated with loops on the surfaces are solved explicitly.

1. Introduction

This paper is devoted to the study of analytic integrable Hamiltonian systems. Integrable systems are important research objects in the theory of dynamical systems, mathematical physics, etc., and they have been studied from various viewpoints. In this paper, we investigate the behaviour of solutions for integrable systems from complex analytic viewpoint. More precisely, we study the structure of analytic continuations of their solutions as functions of complex time.

Let (M, σ) be a complex symplectic manifold of even dimension $2n$, i.e., M a complex manifold and σ a holomorphic two form on M which is closed and nondegenerate. Let $H: M \to \mathbb{C}$ be a meromorphic function on M. It defines a meromorphic Hamiltonian vector field X_H by the relation $\sigma(X_H, \cdot) = dH$. The Poisson bracket of two functions F and G is defined by $\{F, G\} := \sigma(X_F, X_G)$. Let $G_1, \ldots, G_k$ $(k \geq 2)$ be functions which are holomorphic in a domain $\Omega \subset M$. They are said to be *Poisson commuting* (or *in involution*) in Ω if $\{G_i, G_j\} \equiv 0$ in Ω for all $i, j = 1, \ldots, k$. We say that the vector field

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X_H is *integrable* in a domain Ω if it possesses n Poisson commuting integrals $G_1, \dots, G_{n-1}, G_n = H$ which are holomorphic and functionally independent in Ω. Here the n functions $G_1, \dots, G_n$ are said to be functionally independent in Ω if n differentials $dG_1, \dots, dG_n$ are linearly independent on an open and dense subset of Ω.

Let $t \mapsto \phi^t(z) \in M$ be the local orbit of X_H through a point $z \in M$ at $t = 0$ (i.e., $\phi^0(z) = z$), where 'local' means that $\phi^t(z)$ is considered only for sufficiently small $|t| \geq 0$. The orbit of X_H through z is obtained by analytically continuing $\phi^t(z)$ maximally with respect to $t \in \mathbb{C}$, which leads to a so-called complete analytic function of t. It is in general a multi-valued function (mapping) of $t \in \mathbb{C}$ and defines a Riemann surface $\Gamma \subset M$ with the local coordinate t. We are interested in the structure of the Riemann surface Γ, especially in what restrictions are imposed on the fundamental group $\pi_1(\Gamma)$ when the vector field X_H is integrable in a neighbourhood of Γ. In order to approach this problem, we consider the *holonomy representation* of $\pi_1(\Gamma)$ described as follows.

Let p be an arbitrary point on Γ and $\gamma: [0,1] \to \Gamma$ a closed curve based at p (i.e., $\gamma(0) = \gamma(1) = p$). Then there exists a unique curve $\alpha: [0,1] \to \mathbb{C}$ with $\alpha(0) = 0$ such that γ is obtained by the analytic continuation of the local solution $\phi^t(p)$ along α. The analytic continuation along α gives rise to a map Φ^α which takes a point z in a small neighbourhood U of p to the end point of the analytic continuation of $\phi^t(z)$ along α. We note that p is a fixed point of Φ^α. Using this map Φ^α, we now introduce the Poincaré map associated with the loop γ. To this end, we choose a local transverse section Σ to the vector field X_H. Namely, Σ is a local complex submanifold of M of codimension 1 which contains p and is transversal to the vector field X_H. Let π be the projection map which takes a point $z \in U$ to the intersection of $\phi^t(z)$ with Σ. Then we define the *Poincaré map* Ψ^γ by

$$\Psi^\gamma := \pi \circ \Phi^\alpha|\Sigma' : \Sigma' \to \Sigma, \tag{1.1}$$

where Σ' is a subdomain of Σ such that $p \in \Sigma'$ and $\pi \circ \Phi^\alpha(\Sigma') \subset \Sigma$. Moreover, assuming that $\Gamma \subset H^{-1}(h)$ and setting $\Sigma_h := \Sigma \cap H^{-1}(h)$ and $\Sigma'_h := \Sigma' \cap H^{-1}(h)$, we define the *reduced Poincaré map* ψ^γ by

$$\psi^\gamma := \Psi^\gamma|\Sigma'_h : \Sigma'_h \to \Sigma_h . \tag{1.2}$$

It turns out that Σ_h (or Σ'_h) is a symplectic submanifold of dimension $2n-2$ with the symplectic structure $\sigma|\Sigma_h$ and that ψ^γ is a symplectic diffeomorphism around the fixed point p. We can identify Σ_h and p with $\mathbb{C}^{2n-2}$ and the origin $z=0$ respectively and consider the map ψ^γ to be $\psi^\gamma \in \mathrm{Symp}(\mathbb{C}^{2n-2},0)$. Here, $\mathrm{Symp}(\mathbb{C}^{2n-2},0)$ is the group of germs of holomorphic symplectic diffeomorphisms in $\mathbb{C}^{2n-2}$ at the fixed point 0, which is the origin of $\mathbb{C}^{2n-2}$. Finally it is easily seen that $\psi^\gamma = \psi^{\gamma'}$ for any loop γ' homotopic to γ. Therefore we can define a representation of $\pi_1(\Gamma)=\pi_1(\Gamma,p)$ by

$$\rho\colon \pi_1(\Gamma,p) \ni [\gamma] \mapsto \psi^\gamma \in \mathrm{Symp}(\mathbb{C}^{2n-2},0),$$

where $[\gamma]$ is the homotopy class represented by γ. The map ρ gives clearly a homomorphism from $\pi_1(\Gamma,p)$ to $\mathrm{Symp}(\mathbb{C}^{2n-2},0)$ and we call it the *holonomy representation of* $\pi_1(\Gamma,p)$. The image $\rho(\pi_1(\Gamma,p))$ is called the *holonomy group* of the orbit Γ. In §2, we shall discuss in more detail about the holonomy groups.

Our purpose is to give necessary conditions for integrability of the vector field X_H in terms of restrictions on the holonomy groups. Those conditions are concerned with special orbits satisfying the non-resonance condition defined below.

Let $\Gamma \subset H^{-1}(h_0)$ be an orbit of X_H through a point p and γ a loop on Γ based at p. Assume that the linear map $D\psi^\gamma(p)$ has no eigenvalue equal to 1. Here and in what follows, for a mapping f we denote by $Df(p)$ the derivative (linearized mapping) of f at a point p. Then there exists a family of fixed points $p_h \in \Sigma_h$ of Ψ^γ satisfying $p_{h_0}=p$ and depending analytically on the parameter h. These fixed points give rise to a family of orbits Γ_h through p_h and a family of loops $\gamma_h \subset \Gamma_h$, depending analytically on h, with base points p_h such that $\gamma_{h_0}=\gamma$ and $\psi^{\gamma_h}(p_h)=p_h$. Here we assume that the parameter h runs over a neighbourhood $V \subset \mathbb{C}$ of h_0 and denote the family of orbits by $\{\Gamma_h\}_V$ and that of loops by $\{\gamma_h\}_V$. Since a map $f \in \rho(\pi_1(\Gamma_h,p_h))$ is symplectic, the eigenvalues of $Df(p_h)$ occur in pairs λ_i, λ_i^{-1} $(i=1,\dots,n-1)$ (see [1]). We say that the eigenvalues satisfy the non-resonance condition (or the fixed point p_h is said to be non-resonant) if the following condition holds:

$$\prod_{\nu=1}^{n-1} \lambda_\nu^{k_\nu} \neq 1 \quad \text{for all} \quad (k_1,\dots,k_{n-1}) \in Z^{n-1}\setminus\{0\}. \tag{1.3}$$

We say that the loop family $\{\gamma_h\}_V$ is *non-resonant* if the eigenvalues of $D\psi^{\gamma_h}(p_h)$ are all distinct for any $h \in V$ and if they satisfy the non-resonance condition for some $h \in V$. In this case, the fixed points p_h are non-resonant generically for $h \in V$.

We shall show the existence of a system of local coordinates $\varsigma = (\xi, \eta)$ with $\xi, \eta \in \mathbb{C}^{n-1}$ in which

$$p_h = (0,0), \quad \sigma|\Sigma_h = \sum_{i=1}^{n-1} d\xi_i \wedge d\eta_i \tag{1.4}$$

and all elements of the holonomy groups $\rho(\pi_1(\Gamma_h, p_h))$ are in Birkhoff normal form defined as follows:

Definition. (i) Let $G = G(\varsigma)$ be a holomorphic function (i.e., a convergent power series) in a neighbourhood of $\varsigma = 0$. It is said to be in *Birkhoff normal form* if it is actually a function of $n-1$ variables $\xi_i\eta_i$ only.

(ii) A symplectic map $f = f(\varsigma)$ is said to be in *Birkhoff normal form* if it is the time-1 map $\exp X_K$ with Hamiltonian $K = K(\varsigma)$ in Birkhoff normal form.

In the above, the map $\exp X_K$ is obtained by setting $t = 1$ for the flow $\exp(tX_K)$ of the Hamiltonian vector field

$$\dot{\xi}_i = \frac{\partial K}{\partial \eta_i}, \quad \dot{\eta}_i = -\frac{\partial K}{\partial \xi_i} \qquad (i = 1, \ldots, n-1).$$

When K is in Birkhoff normal form, $\omega_i := \xi_i\eta_i$ are integrals of this vector field and $\exp X_K \colon (\xi, \eta) \mapsto (\xi', \eta')$ is written explicitly as

$$\xi_i' = \xi_i \exp(K_{\omega_i}), \quad \eta_i' = \eta_i \exp(-K_{\omega_i}), \quad (i = 1, \ldots, n-1). \tag{1.5}$$

In particular, if the reduced Poincaré map ψ^{γ_h} is in Birkhoff normal form, it has the form

$$\xi_i' = \lambda_i \xi_i + \cdots, \quad \eta_i' = \lambda_i^{-1} \eta_i + \cdots \quad (i = 1, \ldots, n-1),$$

where λ_i, λ_i^{-1} are the eigenvalues of the linear map $D\psi^{\gamma_h}(p_h)$.

Our result is the following

Theorem 1. *Let $\{\Gamma_h\}_V$ be the family of orbits of a Hamiltonian vector field X_H through $p_h \in H^{-1}(h)$ and let $\{\gamma_h\}_V$ be a non-resonant loop family with base points p_h. Assume that the vector field X_H is integrable in a neighbourhood of $\{\Gamma_h\}_V$. Then there exists an open and dense subset $\widehat{V}$ of V with the following property: for any $h \in \widehat{V}$ fixed, there exists, in a* *neighbourhood of p_h in Σ_h, a system of holomorphic local coordinates (ξ, η) satisfying (1.4) in which the reduced Poincaré map ψ^{γ_h} is in Birkhoff normal form and furthermore the following holds for any mapping $f \in \rho(\pi_1(\Gamma_h, p_h))$:*

(i) *The linear map $f_0 := Df(p_h) : (\xi, \eta) \to (\xi', \eta')$ satisfies the condition that for each $i \in \{1, \ldots, n-1\}$ either of the relations*

$$\begin{cases} \xi_i' = c_i \xi_j \\ \eta_i' = c_i^{-1} \eta_j \end{cases} \quad or \quad \begin{cases} \xi_i' = c_i \eta_j \\ \eta_i' = -c_i^{-1} \xi_j \end{cases}$$

holds for some $j \in \{1, \ldots, n-1\}$, where $c_i \in \mathbb{C}$ are constants.

(ii) *Let G be a holomorphic integral of X_H in a neighbourhood of Γ_h. Then the function $G|\Sigma_h$ is in Birkhoff normal form. Moreover it is invariant under the linear map f_0 as well as under f.*

(iii) *The map $\hat{f} := f_0^{-1} \circ f$ is in Birkhoff normal form.*

In the above, we consider the family $\{\Gamma_h\}_V$ as the point set constituting a local complex manifold of dimension 2. Theorem 1 implies the existence of coordinates (ξ, η) in which the nonlinear part of every map $f \in \rho(\pi_1(\Gamma_h, p_h))$ can be solved explicitly. It is a nonlinear version of Ziglin's theorem [12]. The assertion (i) above is the same as Ziglin's theorem. Ziglin considered a linear representation defined by analytic continuations of solutions for the variational equation (linearized equation) along the orbit of X_H. In other words, Ziglin's theorem corresponds to the linear part of our theorem and it does not contain anything about the Birkhoff normal forms. This connection between our result and Ziglin's one will be discussed in more detail in §3 (c). Also our results complement those in [5] which dealt with the behaviour (for real time) of solutions for real analytic integrable Hamiltonian systems. To avoid repetitions, we make use of several technical details that appear in [5]. A main tool is the convergence proof of Birkhoff's normalization in [4], [5].

Although our primary interest is in the structure of orbits of a Hamiltonian vector field, the results can be formulated in more general form concerning orbits of several commuting Hamiltonian vector fields. Here an orbit of several commuting Hamiltonian vector fields will be defined precisely in §2 (a) and it will turn out that the orbit of k commuting vector fields is a k-dimensional complex manifold, which is a generalization of a Riemann surface in the case of an orbit of one vector field. We shall give the definition of the holonomy group for the orbit of k vector fields in §2 (b). In §3, after consideration about the existence of a family of loops (§3 (a)), we shall reformulate the main result as Theorems 2 and 3 which include Theorem 1 as the special case $k = 1$. Being technically different from Theorem 1, Theorems 2 and 3 will be formulated as those asserting the existence of symplectic coordinate system in a neighbourhood (in M) of p_h. The reason why we take the subset $\widehat{V}$ of V in Theorem 1 will be clarified there. Moreover we shall illustrate our results with a simple example in §3 (d). We shall prove Theorems 2 and 3 in §4 and §5.

2. Holonomy groups of orbits of several commuting vector fields

In this section, we will consider an orbit of k commuting Hamiltonian vector fields and will define its holonomy group. Throughout this paper from now on, we let (M, σ) be a complex symplectic manifold of dimension $2n$ and let $H_1, \dots, H_k$ be k meromorphic functions on M. Although the functions $H_1, \dots, H_k$ are assumed to be meromorphic on M, our consideration will be done only in a domain of M where $H_1, \dots, H_k$ are holomorphic. We assume that $H_1, \dots, H_k$ are Poisson commuting and functionally independent in the domain of M.

First we will clarify the meaning of orbits of several commuting vector fields.

(a) *Definition of an orbit of k commuting vector fields*

Let $p \in M$ and assume that $H_1, \dots, H_k$ are holomorphic at p and that k differentials $dH_1, \dots, dH_k$ are linearly independent at p. We denote by ϕ_i^t the flow of the Hamiltonian vector field X_{H_i}. For $t = (t_1, \dots, t_k) \in \mathbb{C}^k$ with $|t| := \max_i |t_i| \geq 0$ sufficiently small, we set

$$\phi^t(p) := \phi_1^{t_1} \circ \cdots \circ \phi_k^{t_k}(p) .$$

Here $\phi^t(p)$ can be represented in a local coordinate system as a power series in $t_1, \dots, t_k$ which is convergent in a polydisk $|t| < \delta$, and it is called the *local solution* of k vector fields $X_{H_1}, \dots, X_{H_k}$ through $p \in M$. Taking the constant δ as large as possible, the polydisk $|t| < \delta$ will be called below the domain of convergence of the local solution $\phi^t(p)$. Since $\phi_1^{t_1}, \dots, \phi_k^{t_k}$ commute in a neighbourhood of p, $\phi^t(p)$ is independent of the order of $\phi_1^{t_1}, \dots, \phi_k^{t_k}$ in their composition.

We define an *orbit* of k vector fields $X_{H_1},\dots,X_{H_k}$ as the point set in M obtained by analytic continuations of this local solution maximally. Analytic continuations of $\phi^t(p)$ are defined along continuous curves on $\mathbb{C}^k$ as follows: Let $\alpha: [0,1] \to \mathbb{C}^k$ be a continuous curve parametrized by s and satisfying $\alpha(0) = 0 \in \mathbb{C}^k$. Then we can continue analytically the local solution $\phi^t(p)$ along α in the following way. If $|\alpha(s)| \geq 0$ is sufficiently small, we define the point $\varphi(\alpha(s);p)$ by

$$\varphi(\alpha(s);p) := \phi^{\alpha(s)}(p).$$

Next, let $0 < s_1 < 1$ and assume that $\beta_1 = \alpha(s_1)$ is contained in the domain of convergence of $\phi^t(p)$. Then one can consider the local solution of $X_{H_1},\dots,X_{H_k}$ through $\phi^{\beta_1}(p)$ and therefore we define the point $\varphi(\alpha(s);p)$ for $s > s_1$ by $\varphi(\alpha(s);p) := \phi^{\alpha(s)-\beta_1}(\phi^{\beta_1}(p))$. Further we can define inductively the point $\varphi(\alpha(s);p) \in M$ by

$$\varphi(\alpha(s);p) = \phi^{\beta_N} \circ \phi^{\beta_{N-1}} \circ \cdots \circ \phi^{\beta_1}(p) \tag{2.1}$$

with

$$\beta_i := \alpha(s_i) - \alpha(s_{i-1}), \qquad 0 = s_0 < s_1 < s_2 < \cdots < s_N = s,$$

where $H_1,\dots,H_k$ are assumed to be holomorphic at points $\varphi(\alpha(s_i);p)$ and for every $i = 1,\dots,N$, β_i is contained in the domain of convergence of the local solution through $\phi^{\beta_{i-1}} \circ \cdots \circ \phi^{\beta_1}(p)$. If this procedure is possible up to $s = 1$, it is called the analytic continuation of the local solution $\phi^t(p)$ along α. Let

$$C(p) := \{\alpha: [0,1] \to \mathbb{C}^k \mid \alpha = \alpha(s) \text{ is continuous in } s \in [0,1],\ \alpha(0) = 0 \\ \text{and } \varphi(\alpha(s);p) \text{ is well defined for all } 0 \leq s \leq 1\}.$$

Then the orbit Γ of k vector fields $X_{H_1},\dots,X_{H_k}$ through p is defined as

$$\Gamma := \bigcup_{\alpha \in C(p)} \{\varphi(\alpha(s);p) \in M \mid 0 \leq s \leq 1\}.$$

Here Γ is independent of p in the following sense: If p' is an arbitrary point on Γ, then the analytic continuation of the local solution through p' gives rise to the same orbit as Γ. For every point $z \in \Gamma$, there exists a unique local solution $\phi^t(z)$ for $t \in \mathbb{C}^k$ with $|t| \geq 0$ sufficiently small. Since $\phi^t(z)$ gives a one-to-one correspondence between a neighbourhood of z in Γ and a neighbourhood of $t = 0$ in $\mathbb{C}^k$, the orbit Γ is a k-dimensional complex manifold with the local coordinates

$t = (t_1, \ldots, t_k)$. One can easily see that $dH_1, \ldots, dH_k$ are linearly independent at every point on Γ (see [[5], Proposition 2.1]).

(b) *Definition of the holonomy group*

Let us now define the holonomy group of the orbit Γ of k vector fields defined above. In what follows, we let p be an arbitrary point on Γ. We assume that

$$\Gamma \subset H^{-1}(h) \ ; \ \ H := (H_1, \ldots, H_k), \quad h \in \mathbf{C}^k.$$

To any closed curve (loop) $\gamma: [0,1] \to \Gamma$ based at p (i.e., $\gamma(0) = \gamma(1) = p$), there corresponds uniquely a curve $\alpha: [0,1] \to \mathbf{C}^k$ with $\alpha(0) = 0$ such that γ is parametrized as

$$\gamma(s) = \varphi(\alpha(s); p) \quad (s \in [0,1]).$$

The analytic continuation along α defines a mapping

$$\Phi^\alpha: z \mapsto \varphi(\alpha(1); z).$$

Here z is assumed to be in a small neighbourhood of p and $\varphi(\alpha(1); z)$ is the end point of the analytic continuation of the local solution $\phi^t(z)$ along α. Since the vector fields X_{H_i} are hamiltonian, Φ^α is symplectic, i.e., $(\Phi^\alpha)^*\sigma = \sigma$. In fact, if we set $\beta_i = (\beta_i^1, \beta_i^2, \ldots, \beta_i^k)$ in the expression (2.1), the map ϕ^{β_i} $(= \phi_1^{\beta_i^1} \circ \cdots \circ \phi_k^{\beta_i^k})$ can be expressed as

$$\phi^{\beta_i} = \exp X_{\widehat{H}_i} \quad \text{with} \quad \widehat{H}_i = \sum_{\nu=1}^{k} \beta_i^\nu H_\nu$$

since the vector fields $X_{H_1}, \ldots, X_{H_k}$ commute locally. Here the time 1 maps ϕ^{β_i} are symplectic and therefore the composition Φ^α is symplectic.

Let Σ be a local complex submanifold of M of (complex) codimension k such that $\Sigma \ni p$ and

$$T_pM = T_p\Sigma \oplus \operatorname{span}(X_{H_1}(p), \ldots, X_{H_k}(p)). \tag{2.2}$$

We call Σ a *transverse section* to the k vector fields $X_{H_1}, \ldots, X_{H_k}$ at p. By the transversality (2.2), for a small neighbourhood U of p we can define the projection map π by

$$\pi: U \ni z \mapsto \pi(z) := \Sigma \cap \{\phi^t(z) \mid \phi^t(z) \in U \ \ (|t|: \text{small})\}.$$

Then in the same way as in §1 we can define the Poincaré map Ψ^γ and the reduced Poincaré map ψ^γ by (1.1) and (1.2) respectively. Here $\Sigma_h = \Sigma \cap H^{-1}(h)$ with $H = (H_1, \dots, H_k)$ is a symplectic submanifold of dimension $2n - 2k$ with the symplectic structure $\sigma|\Sigma_h$ and the map ψ^γ gives a symplectic diffeomorphism around a fixed point p (see [[5], Lemma 3.1]). Consequently the map Ψ^γ can be considered as a family of parametrized symplectic diffeomorphisms.

In the above, the choice of Σ' depends on the loop γ. For example, if we consider a composition γ^n with large n in place of γ, then Σ' would have to be chosen very small. By this reason, we consider the mappings Ψ^γ and ψ^γ as $\Psi^\gamma \in \mathrm{Diff}(\Sigma, p)$ and $\psi^\gamma \in \mathrm{Symp}(\Sigma_h, p)$ respectively. Here, $\mathrm{Diff}(\Sigma, p)$ is the group of germs of holomorphic diffeomorphisms in Σ at the fixed point p, and $\mathrm{Symp}(\Sigma_h, p)$ is the group of germs of holomorphic symplectic diffeomorphisms in Σ_h at the fixed point p. In what follows, we will use the same notation as these to denote groups of germs of holomorphic (symplectic) diffeomorphisms at a fixed point. Introducing local coordinates, we can identify the groups $\mathrm{Diff}(\Sigma, p)$ and $\mathrm{Symp}(\Sigma_h, p)$ with $\mathrm{Diff}(\mathbb{C}^{2n-k}, 0)$ and $\mathrm{Symp}(\mathbb{C}^{2n-2k}, 0)$ respectively, where we consider $\mathbb{C}^{2n-2k}$ as a symplectic manifold with the standard symplectic structure $\sigma = \sum_{i=1}^{n-k} d\xi_i \wedge d\eta_i$. The mappings Ψ^γ and ψ^γ are independent of the choice of the transverse section Σ. In fact, if we introduce a special coordinate system (u, v, ξ, η) satisfying (3.1) below, then one can easily see that the expressions of Ψ^γ and ψ^γ in this coordinate system are independent of the choice of Σ.

If $\gamma' \subset \Gamma$ is another loop homotopic to γ with the base point p, the corresponding curve $\alpha' \subset \mathbb{C}^k$ is also homotopic to α with the same endpoints. From the local single-valuedness of the solutions it follows that $\Psi^\gamma = \Psi^{\gamma'}$ and hence $\psi^\gamma = \psi^{\gamma'}$. We define a representation of $\pi_1(\Gamma, p)$ by

$$\rho \colon \pi_1(\Gamma, p) \ni [\gamma] \mapsto \psi^\gamma \in \mathrm{Symp}(\Sigma_h, p).$$

Clearly the map ρ gives a homomorphism from $\pi_1(\Gamma, p)$ to $\mathrm{Symp}(\Sigma_h, p)$. The map ρ is called a *holonomy representation* of $\pi_1(\Gamma, p)$ (or of Γ) and the image $\rho(\pi_1(\Gamma, p))$ is called the *holonomy group* of Γ. The holonomy group is independent of the choice of the base point p as well as that of the transverse section Σ in the following sense : Let $\hat{p} \in \Gamma$ and $\widehat{\Sigma}$ $(\ni \hat{p})$ be another pair of a base point and a transverse section and let $\hat{\rho} \colon \pi(\Gamma, \hat{p}) \to \mathrm{Symp}\,(\widehat{\Sigma}_h, \hat{p})$ be a representation which is defined in the same way as ρ. Then $\pi_1(\Gamma, p)$ and $\pi_1(\Gamma, \hat{p})$ are isomorphic and we note that there exists a curve $\alpha \colon [0, 1] \to \mathbb{C}^k$ connecting p and $\hat{p}$. The map

Φ^{α} induces a local symplectic diffeomorphism from a neighbourhood of p in Σ_h into a neighbourhood of $\hat{p}$ in $\hat{\Sigma}_h$. Therefore the holonomy groups $\rho(\pi_1(\Gamma, p))$ and $\hat{\rho}(\pi_1(\Gamma, \hat{p}))$ are conjugate via the local diffeomorphism .

3. Statement of the results

We will state the results for orbits of k commuting vector fields. The main results will be formulated as Theorems 2 and 3 in subsection (**b**) and it will turn out that Theorem 1 is a corollary of Theorems 2 and 3. We will use the same notation as in the previous section.

(**a**) *Existence of a family of fixed points of Ψ^{γ} and a special local coordinate system*

Let Γ be the orbit of k commuting vector fields $X_{H_1}, \ldots, X_{H_k}$ through a point p and assume that

$$\Gamma \subset H^{-1}(h_0) \quad \text{with} \quad h_0 \in \mathbb{C}^k.$$

Let us consider the Poincaré map Ψ^{γ} associated with a loop $\gamma \in \Gamma$ based at p.. We first claim that fixed points of Ψ^{γ} in Σ form a holomorphic curve under a nondegeneracy condition.

Proposition 1. *Assume that the linear map $D\psi^{\gamma}(p)$ has no eigenvalue equal to 1. Then there exists a unique family of fixed points $p_h \in \Sigma_h$ of Ψ^{γ} such that $p_{h_0} = p$ and p_h depends on h close to h_0 analytically.*

This is easily proved. In fact, let us consider the equation $\Psi^{\gamma}(z) = z$ in a neighbourhood of p and note that $\det(D\psi^{\gamma}(p) - I) \neq 0$ under the assumption. Then, by the implicit function theorem we obtain a unique family of fixed points $z = p_h$ of Ψ^{γ} stated above. □

We can easily see that there exists a curve α_h: $[0,1] \to \mathbb{C}^k$ depending on h continuously and satisfying

$$\alpha_h(0) = 0, \quad \alpha_{h_0} = \alpha, \quad \Phi^{\alpha_h}(p_h) = p_h.$$

Hence there exists a family of closed curves

$$\gamma_h: s \mapsto \varphi(\alpha_h(s); p_h) \quad (s \in [0,1]).$$

The statement of Proposition 1 is local. Namely the parameter h is restricted to a small neighbourhood V_{h_0} of h_0. Applying Proposition 1 to the map Ψ^{γ_h} with $h \in V_{h_0}$, we can also continue the family of curves $\{\alpha_h\}$ (or loops $\{\gamma_h\}$) to a family with parameter h running over a neighbourhood of h_0 larger than V_{h_0}. We denote by V the neighbourhood of h_0 where the parameter h runs over and denote by $\{\gamma_h\}_V$ the family of loops γ_h. However the size of V will not be important for our purpose. Therefore we will not necessarily take V as large as possible for the formulation of our main results.

Definition. Let Γ be an orbit of k vector fields $X_{H_1}, \dots, X_{H_k}$. A loop $\gamma \subset \Gamma$ is said to be *nondegenerate* if the assumption of Proposition 1 is satisfied. The family of loops $\{\gamma_h\}_V$ described above is called a *nondegenerate loop family with base points* p_h.

Let $\{\gamma_h\}_V$ be a nondegenerate loop family with base points p_h and let Γ_h be the orbit of $X_{H_1}, \dots, X_{H_k}$ through p_h. We have a family of orbits $\{\Gamma_h\}_V := \{\Gamma_h : h \in V\}$ such that $\gamma_h \subset \Gamma_h$. For any $h \in V$ fixed, we consider the holonomy representation

$$\rho: \pi_1(\Gamma_h, p_h) \to \mathrm{Symp}(\Sigma_h, p_h).$$

The definition of the representation ρ implies that $\rho([\gamma_h]) = \psi^{\gamma_h} (= \Psi^{\gamma_h}|\Sigma_h)$. Our results will be concerned with the holonomy groups $\rho(\pi_1(\Gamma_h, p_h))$ in the case when the vector fields $X_{H_1}, \dots, X_{H_k}$ are integrable, and the loop family $\{\gamma_h\}_V$ will play a key role for their formulation.

Let us consider mappings belonging to $\mathrm{Diff}(\Sigma, p_h)$ or $\mathrm{Symp}(\Sigma_h, p_h)$ in a special coordinate system. For any $h \in V$ fixed, it is easily verified that in a neighbourhood of p_h there exists a system of local symplectic coordinates (u, v, ξ, η) with $u, v, \in \mathbf{C}^k$ and $\xi, \eta \in \mathbf{C}^{n-k}$ such that

$$H = v, \quad \sigma = \sum_{i=1}^{k} du_i \wedge dv_i + \sum_{i=1}^{m} d\xi_i \wedge d\eta_i \qquad (m = n - k) \tag{3.1}$$

and

$$p_v = (0, v, 0, 0). \tag{3.2}$$

For the proof of this fact, we refer to the proof of [[5], Proposition 4.1]. In this coordinate system, we define the transverse section Σ by

$$\Sigma = \{(u, v, \xi, \eta) \in U \mid u = 0\},$$

where U is a neighbourhood (in $\mathbb{C}^{2n}$) of $p_h = (0,h,0,0)$. Then Σ_h is given as

$$\Sigma_h = \{(u,v,\xi,\eta) \in U \mid u=0,\ v=h\}.$$

Let G be an integral of $X_{H_1},\dots,X_{H_k}$ and assume that it is holomorphic in a neighbourhood of p_h. Then we have identities $\{G,H_i\} \equiv 0$, equivalently $\partial G/\partial u_i \equiv 0$. Hence G is a function of v, ξ, η and we write it as $G = G(v,\varsigma)$ with $\varsigma = (\xi,\eta)$. We consider the Poincaré map Ψ^{γ_h} as $\Psi^{\gamma_h} \in \mathrm{Diff}(\mathbb{C}^{2n-k}, z_h)$, where $z_h = (h,0)$ with coordinates (v,ς). Then Ψ^{γ_h} can be written as

$$\Psi^{\gamma_h}(v,\varsigma) = (\,v, \Psi_v^{\gamma_h}(\varsigma)\,),$$

where $\Psi_v^{\gamma_h} := \Psi^{\gamma_h}|\Sigma_v' \colon \Sigma_v' \to \Sigma_v$ and it is a symplectic diffeomorphism . We note that p_v is a fixed point of $\Psi_v^{\gamma_h}$ for any v. More generally, for any loop γ on Γ with the base point p_h the Poincaré map $F := \Psi^{\gamma}$ can be written as

$$F(v,\varsigma) = (\,v, f(v,\varsigma)\,), \tag{3.3}$$

where $f(v,\varsigma) \in \mathbb{C}^{2n-2k}$ is a holomorphic vector function of v and ς in a neighbourhood of $(v,\varsigma) = (h,0)$ such that $f(h,0) = 0$. It is to be noted that $f(v,0) \neq 0$ for $v \neq h$ in general, namely p_v is not necessarily a fixed point of Ψ^{γ}.

The representation ρ can be viewed as

$$\rho\colon \pi_1(\Gamma_h, p_h) \to \mathrm{Symp}(\mathbb{C}^{2n-2k}, 0)$$

and any map $f \in \rho(\,\pi_1(\Gamma_h,p_h)\,)$ can be written as $\varsigma' = f(\varsigma)$ defined in a neighbourhood of a fixed point $\varsigma = 0$. For $f \in \rho(\,\pi_1(\Gamma_h,p_h)\,)$, let us decompose it into its linear and nonlinear parts as follows :

$$f = f_0 \circ \hat{f}\,; \quad f_0(\varsigma) = Df(0)\,\varsigma\,, \quad \hat{f}(\varsigma) = \varsigma + O(|\varsigma|^2)\,. \tag{3.4}$$

Here $Df(0) = Df(p_h)$ in the previous notation and $O(|\varsigma|^2)$ denotes a vector function of ς whose components are convergent power series in ς containing only terms of order ≥ 2.

By using the above coordinate system, we can prove

Proposition 2. *Let $\Gamma \subset H^{-1}(h)$ be an orbit of k vector fields $X_{H_1},\dots,X_{H_k}$ containing a nondegenerate loop. Then, for any integral G of $X_{H_1},\dots,X_{H_k}$ which is holomorphic in a neighbourhood of Γ, its derivative dG is linearly dependent on $dH_1,\dots,dH_k$ at every point on the orbit Γ.*

Proof. Let γ be a nondegenerate loop on Γ and ψ^γ the reduced Poincaré map associated with γ. Let p be an arbitrary point on γ and (u, v, ξ, η) coordinates in a neighbourhood of p satisfying (3.1) and (3.2). By assumption, $G = G(h, \varsigma)$ is invariant under ψ^γ and therefore $G(h, \psi^\gamma(\varsigma)) = G(h, \varsigma)$. Noting that $\psi^\gamma(0) = 0$, the differentiation of this identity with respect to ς yields

$$^t(D\psi^\gamma(0))\frac{\partial G}{\partial \varsigma}(h,0) = \frac{\partial G}{\partial \varsigma}(h,0),$$

where $^t(\,)$ denotes the transposition of a matrix. If $\partial G/\partial\varsigma(h,0) \neq 0$, this implies that $D\psi^\gamma(0)$ has an eigenvalue equal to 1. It contradicts the assumption that γ be nondegenerate. Therefore the following relation holds at the point p:

$$dG = \sum_{i=1}^{k} c_i dH_i \qquad \text{for some vector } c = (c_1, \dots, c_k) \in \mathbb{C}^k. \tag{3.5}$$

Since $X_{H_i} = \partial/\partial u_i$ and G is independent of u in the neighbourhood of p, the relation (3.5) holds with the fixed constant vector c along the part of Γ contained in the neighbourhood of p. Moreover by successive use of the same coordinate system as above, one can see that the relation (3.5) holds along the whole orbit Γ. This completes the proof. □

Let $\{\gamma_h\}_V$ be a nondegenerate loop family and $\{\Gamma_h\}_V$ the family of orbits such that $\gamma_h \subset \Gamma_h$. Then it follows from Proposition 2 that for any integral G of $X_{H_1}, \dots, X_{H_k}$ the relation (3.5) holds everywhere on the family $\{\Gamma_h\}_V$, where the constant vector c depends only on h. In view of integrable systems, this indicates a special property of the orbits considered above.

(b) *Main results*

To state the results, we begin by preparing some definitions. Let us consider the loop family $\{\gamma_h\}_V$ associated with a nondegenerate loop $\gamma = \gamma_{h_0}$.

Definition. A nondegenerate loop family $\{\gamma_h\}_V$ with base points p_h is called a *non-resonant loop family* if the eigenvalues of $D\psi^{\gamma_h}(p_h)$ are all distinct for any $h \in V$ and if they satisfy the non-resonance condition for some $h \in V$.

Here, since the map ψ^{γ_h} is symplectic, the eigenvalues of $D\psi^{\gamma_h}(p_h)$ occur in pairs λ_i, λ_i^{-1} ($i = 1, \dots, m$; $m = n - k$) (see [1]). We say that the eigenvalues satisfy the non-resonance condition (or we say that the fixed point p_h is non-resonant) if condition (1.3) holds with $n - 1$ replaced by m. For a non-resonant loop family, the fixed point p_h is non-resonant for any h belonging to a residual subset of V (see [[5], Lemma 5.2]). In other words, the fixed points are generically non-resonant.

Our results will be obtained through normalization of the map Ψ^{γ_h}. It is well known that if the eigenvalues of the derivative $D\Psi_v^{\gamma_h}(0)$ satisfy the non-resonance condition, there exists a formal symplectic transformation which takes $\Psi_v^{\gamma_h}(= \Psi^{\gamma_h}(v,\cdot))$ into Birkhoff normal form. To prove the convergence of this transformation, we involve the so-called small divisor problem and we have divergence in general. Furthermore it is not possible in resonant case to find the transformation even formally. However in our situation associated with integrable systems, we can prove the existence of a convergent symplectic transformation which takes $\Psi_v^{\gamma_h}$ into Birkhoff normal form for any v. This is the content of Theorem 2 below. In order to state the results, we introduce the following definition.

Definition. (i) A holomorphic function $G = G(v,\varsigma)$ is said to be in *parametrized normal form* if it is in Birkhoff normal form for each v fixed.

(ii) A map $F \in \mathrm{Diff}(\mathbb{C}^{2n-k}, z_h)$ is said to be in *parametrized normal form* if it has the form (3.3) with $f(v,\varsigma)$ being in Birkhoff normal form for each v fixed.

We introduce a further definition. Let $G(v,\varsigma)$ be a convergent power series in ς around $\varsigma = 0$ with coefficients holomorphic in v. It can be written in the form

$$G = G(v,0) + G^0(v,\varsigma) + G^1(v,\varsigma) + \cdots ; \quad G^0(v,\varsigma) \not\equiv 0, \tag{3.6}$$

where $G^d(v,\varsigma)$ $(d = 0,1,\dots)$ are homogeneous polynomials in ς of degree $s+d$ with coefficients holomorphic in v. The integer $s \geq 1$ is the degree of the polynomial $G^0(v,\varsigma)$ in ς. We call the function $G^0(v,\varsigma)$ the *lowest order part* of G.

To obtain the result, we assume that there exist m functions G_i $(i=1,\dots,m)$ holomorphic in a neighbourhood of $\{\Gamma_h\}_V$ such that the following two conditions are satisfied:

[C.1] $\{H_i, G_j\} \equiv 0$ for all i,j ;

[C.2] In a neighbourhood of p_h with coordinates (u,v,ξ,η) satisfying (3.1) and (3.2), the lowest order parts of $G_i = G_i(v, \varsigma)$ $(i = 1, \ldots, m)$ are functionally independent functions of ς for $v = h$ fixed.

Here, condition [C.1] implies that G_i are integrals of $X_{H_1}, \ldots, X_{H_k}$. Moreover condition [C.2] includes that $v_1, \ldots, v_k$ and $G_1, \ldots, G_m$ are functionally independent functions of v and ς and consequently $H_1, \ldots, H_k$ and $G_1, \ldots, G_m$ are functionally independent in a neighbourhood of $\{\Gamma_h\}_V$. Conversely it will turn out that the functional independence of $H_1, \ldots, H_k$ and $G_1, \ldots, G_m$ almost implies condition [C.2] (see Proposition 3 below).

Now our results are stated in the following two theorems.

Theorem 2. *Let $\{\Gamma_h\}_V$ be the family of orbits of k commuting vector fields $X_{H_1}, \ldots, X_{H_k}$ through $p_h \in H^{-1}(h)$ and let $\{\gamma_h\}_V$ be a non-resonant loop family with base points p_h. Assume that there exist $m(= n - k)$ functions G_i $(i = 1, \ldots, m)$ holomorphic in a neighbourhood of $\{\Gamma_h\}_V$ and satisfying conditions* [C.1] *and* [C.2] *for some $h \in V$ fixed. Then, in a neighbourhood of p_h there exists a system of holomorphic local coordinates (u, v, ξ, η) with $u, v \in \mathbb{C}^k$, $\xi, \eta \in \mathbb{C}^m$ in which the following holds together with* (3.1) *and* (3.2) :

(i) *The Poincaré map Ψ^{γ_h} is in parametrized normal form.*

(ii) *Any holomorphic integral of $X_{H_1}, \ldots, X_{H_k}$ in a neighbourhood of Γ_h is in parametrized normal form.*

Next we consider arbitrary mappings belonging to the holonomy groups $\rho(\pi_1(\Gamma_v, p_v))$ with v sufficiently close to h.

Theorem 3. *Assume the same hypothesis as in Theorem 2 to be satisfied. Let $v \in V$ be fixed arbitrarily in a sufficiently small neighbourhood of h. Then, in the coordinates introduced in Theorem 2, the following holds for any mapping $f \in \rho(\pi_1(\Gamma_v, p_v))$:*

(i) *The linear map $f_0 := Df(p_v) : (\xi, \eta) \to (\xi', \eta')$ satisfies the condition that for each $i \in \{1, \ldots, n-1\}$ either of the relations*

$$\begin{cases} \xi'_i = c_i \xi_j \\ \eta'_i = c_i^{-1} \eta_j \end{cases} \quad \text{or} \quad \begin{cases} \xi'_i = c_i \eta_j \\ \eta'_i = -c_i^{-1} \xi_j \end{cases} \tag{3.7}$$

holds for some $j \in \{1, \ldots, n-1\}$, where $c_i \in \mathbb{C}$ are constants.

(ii) *Let G be a holomorphic integral of $X_{H_1}, \dots, X_{H_k}$ in a neighbourhood of Γ_v. Then the function $G|\Sigma_v = G(v, \cdot)$ is invariant under the linear map f_0 as well as under f.*

(iii) *The map $\hat{f} := f_0^{-1} \circ f$ is in Birkhoff normal form.*

Remarks. (i) In the above, it is not assumed that $G_1, \dots, G_m$ are in involution. However that condition is hidden behind the assumption that the family $\{\gamma_h\}_V$ be non-resonant. Actually the assertion (ii) of Theorem 2 implies that they are in involution. Hence the vector fields X_{H_i} are integrable in a neighbourhood of $\{\Gamma_h\}_V$ under the assumptions of Theorem 2.

(ii) The assertion (i) of Theorem 3 is independent of coordinates. Namely we can also claim as follows: Let E_i and F_i be the eigenspaces of $D\psi^{\gamma_v}(p_v)$ corresponding to the eigenvalues λ_i and λ_i^{-1} $(i = 1, \dots, m)$ respectively. Then the linear map $A := Df(p_v)$ satisfies either of the following condition (a) or (b).

(a) $AE_i = E_j$ and $AF_i = F_j$ for some $j \in \{1, \dots, m\}$,

(b) $AE_i = F_j$ and $AF_i = E_j$ for some $j \in \{1, \dots, m\}$.

(iii) Let $F := \Psi^\gamma \in \mathrm{Diff}(\Sigma, p_h)$ be the Poincaré map associated with a loop γ. Then it follows from Theorem 5.6 of [5] that F is also in parametrized normal form if F and Ψ^{γ_h} commute. (In [5], we called it simply 'normal form' in place of 'parametrized normal form' in the present paper.) This is a special case of Theorem 3.

Theorem 1 is a corollary to Theorems 2 and 3. It is due to the following

Proposition 3. *Let $G_1, \dots, G_m$ be holomorphic functions in a neighbourhood of $\{\Gamma_h\}_V$ satisfying two conditions* [C.1] *and*

[C.3] *$H_1, \dots, H_k$ and $G_1, \dots, G_m$ are functionally independent.*

Then for any $h \in V$ fixed, there exist a neighbourhood $V_h (\subset V)$ of h and m functions $\widehat{G}_1, \dots, \widehat{G}_m$ holomorphic in a neighbourhood of $\{\Gamma_v\}_{V_h}$ such that the following condition holds together with [C.1] *(for G_i replaced by $\widehat{G}_i$) :*

[C.4] *There exists an open and dense subset $\widehat{V}_h$ of V_h such that the lowest order parts $\widehat{G}_i^0(v, \varsigma)$ $(i = 1, \dots, m)$ are functionally independent* *functions of* ς *for any* $v \in \widehat{V}_h$ *fixed.*

We will prove this proposition in the next section. If we assume that $G_1, \dots, G_m$ satisfy conditions [C.1] and [C.3], then by Proposition 3 there exists an open and dense subset $\widehat{V}$ of V such that condition [C.2] holds for any $h \in \widehat{V}$ and hence the conclusions of Theorems 2 and 3 hold. In particular, Theorem 1 follows from Theorems 2 and 3.

As we stated after Proposition 2, the family of orbits $\{\Gamma_h\}_V$ has the special property that

$$\operatorname{rank}(dH_1, \dots, dH_k, dG_1, \dots, dG_m) = k \quad \text{on} \quad \{\Gamma_h\}_V,$$

where $H_1, \dots, H_k$ and $G_1, \dots, G_m$ are n integrals of $X_{H_1}, \dots, X_{H_k}$ given in the assumption of Theorem 2. This implies that the family $\{\Gamma_h\}_V$ forms a complex manifold of dimension $2k$ which constitutes singularities of rank k of the map $F = (H_1, \dots, H_k, G_1, \dots, G_m): \Omega \to \mathbb{C}^n$, where Ω is a neighbourhood of $\{\Gamma_h\}_V$. Theorem 2 shows, roughly speaking, that for integrable systems local diffeomorphisms defined by analytic continuations near such singularities are essentially determined by their linear parts.

(c) *Connection between Theorem 1 and Ziglin's theorem*

We briefly discuss the connection between our results and Ziglin's theorem [12] (see also [6]). Our results can be considered as a nonlinear version of Ziglin's theorem dealing with monodromy representation of one orbit Γ for a Hamiltonian vector field X_H. The Riemann surface Γ is assumed to contain a loop γ such that the fixed point p of the associated reduced Poincaré map is non-resonant. More precisely, Ziglin did not consider the Poincaré map but analytic continuations of solutions for the (reduced) normal variational equation (linearized equation) along Γ. The analytic continuations are associated with loops on Γ and induce transitions between the systems of fundamental solutions. It gives rise to a linear representation $\tilde{\rho}: \pi_1(\Gamma, p) \to GL(2n-2, \mathbb{C})$. The image $\tilde{\rho}(\pi_1(\Gamma, p))$ is called the monodromy group and its element is called a monodromy matrix. The matrix $\tilde{\rho}([\gamma])$ corresponds to the linear mapping $D\psi^\gamma$ $(\psi^\gamma = \rho([\gamma]))$ in our case, and the eigenvalues of $\tilde{\rho}([\gamma])$ are the same as those of $D\psi^\gamma$.

Ziglin's theorem gives a necessary condition for the existence of meromorphic integrals of X_H under the assumption that the eigenvalues of some monodromy matrix satisfy the non-resonance condition. The necessary condition is the same as condition (i) of Theorem 1. Conditions (ii) and (iii) are new ones which are not contained in Ziglin's theorem. As we shall see in the proof (§4), condition (i) is obtained from the invariance under f_0 of the lowest order parts $G_i^0(v,\varsigma)$. Condition (ii) implies that all homogeneous parts in the expansion of G_i are invariant under the linear map $Df(p_h)$. It means strong restrictions on $Df(p_h)$ in addition to (i).

Clearly our main result, Theorem 1, can be considered a generalization of Ziglin's theorem. However there are some distinctions between the formulation of our result and that of Ziglin. Namely, Ziglin's theorem is concerned with one orbit Γ containing a loop ν for which the *non-resonance condition* holds. Our result is concerned with a family of orbits $\{\Gamma_h\}_V$, stating a necessary condition for the existence of *holomorphic* (not meromorphic) integrals for any $h \in \hat{V}$, $\hat{V}$ being an open and dense subset of V. Notice that there exists in general a dense subset $\hat{V}'$ of $\hat{V}$ such that for $h \in \hat{V}'$, p_h is a resonant fixed point of ψ^{ν_h}, i.e. condition (1.3) fails. Nevertheless, our result implies that an assertion similar to that of Ziglin's theorem holds true also for these parameter values.

Ziglin's theorem is useful for proving non-integrability of specific systems, and there appeared many papers concerning its application (see survey papers [**7**, **11**] and also [**2**, **3**, **9**, **10**, **13**] for details). In order to apply Ziglin's theorem, it is a key point to find a special solution which can be written more or less explicitly (such as elliptic functions, etc.) so that the eigenvalues of monodromy matrices can be known via the corresponding variational equation. Since we deal with more general non-linear monodromy, similar applications seem more difficult in this case.

(d) *An example*

Finally we will give an example to illustrate our results. Let us consider a Hamiltonian system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2) \tag{3.8}$$

with Hamiltonian of the form

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) + U(q_1, q_2);$$
$$U(q_1, q_2) = \frac{1}{2}(Aq_1^2 + Bq_2^2) + Cq_1^2 q_2 + \frac{1}{3}Dq_2^3,$$

where A, B, C, D are real constants. It depends on the values of A, B, C, D whether this system is integrable or not. For example, a criterion based on Ziglin's theorem shows that if $A = B(\neq 0)$, the system is integrable in the whole phase space $\mathbb{C}^4$ only when $C/D = 0,\ 1/6,\ 1/2,\ 1$ (see [3]). Among them, the cases $C/D = 0,\ 1/6,\ 1$ are actually known to be integrable (see [8]). The cases $C/D = 0,\ 1$ are trivial ones which are reduced to separation of variables by a linear symplectic transformation. Let us consider the nontrivial case $C/D = 1/6$, for which the system is integrable for any choice of A and B ($A \neq B$ in general) with an additional integral

$$G(q,p) = (4A-B)(p_1^2+Aq_1^2)+4C\{Aq_1^2q_2-p_1(p_1q_2-p_2q_1)\}+C^2(q_1^4+4q_1^2q_2^2).$$

In what follows, we will illustrate Theorem 1 with this simple example. We note that, if we set $q_1 = p_1 = 0$, the system (3.8) is reduced to the system with one degree of freedom. Its phase curve is given by

$$\frac{1}{2}p_2^2 + U(0, q_2) = h, \tag{3.9}$$

where h is the energy parameter. This is solved for q_2 as an elliptic function of $t \in \mathbb{C}$. To illustrate the non-resonance loop family, let us assume that $B > 0$. The origin $(q,p) = (0,0)$ is an equilibrium point with the characteristic exponents $\pm\sqrt{-A}$ and $\pm\sqrt{-B}$. Therefore the solutions (3.9) with $q_1 = p_1 = 0$ give rise to a family of real periodic orbits $\gamma_h \in H^{-1}(h)$ for $t \in R$ bifurcating from the origin, whose periods tend to $2\pi/\sqrt{B}$ as $h \to +0$. This family corresponds to the one whose existence is established by Liapunov's theorem [1] provided that $\sqrt{A/B}$ is not an integer.

In view of our theory, it defines a loop family $\{\gamma_h\}_V$ with $\gamma_h \subset \Gamma_h \subset H^{-1}(h)$, where Γ_h is the analytic continuation of γ_h. Here we may consider V to be extended to a complex domain. If $\sqrt{A/B}$ is not a rational number, it is a non-resonant loop family because the Floquet multiplies of γ_h converge to $\exp(2\pi i\sqrt{B/A})$ as $h \to +0$ (see [[3], Proposition 1]). Hence Theorem 1 is applicable. In this case, the orbits (3.9) are doubly periodic function of t and the Riemann surfaces Γ_h are homeomorphic to a punctured tori of dimension 2 (with one point removed which corresponds to the pole in the t-plane of the solution). Ziglin's theorem, i.e., the assertion (i) of Theorem 1, implies that the linear parts of the reduced Poincaré maps associated with the two independent periods, one of which is that of γ_h, commute or permute (see [[3], Theorem 1]). Also, the assertions (ii) and (iii) of Theorem 1 are valid for any $h \in \widehat{V}$, $\widehat{V}$ being an open and dense subset of V.

4. Proof of Theorem 2

Theorem 2 is already proved in [[5], Theorem 5.6] in different situation. In this section, we give a brief review of its proof. It plays a role of preliminaries for the proof of Theorem 3. Also we will prove Proposition 3 at the end of this section.

Let $\{\gamma_h\}_V$ be the non-resonant loop family given in the assumption of Theorem 2. We consider all functions locally in the coordinates (u, v, ξ, η) introduced in §3 (a) in a neighbourhood U of $p_h = (0, h, 0, 0)$. Let us expand functions $G_i = G_i(v, \varsigma)$ in the form (3.6) and assume that the coefficients are holomorphic functions of v in a sufficiently small neighbourhood V_h of $v = h$.

For any $v \in V_h$, $G_i|\Sigma_v = G_i(v, \cdot)$ are integrals of the reduced Poincaré map ψ^{γ_v} and hence we have

$$G_i(v, \psi^{\gamma_v}(\varsigma)) = G_i(v, \varsigma).$$

Then comparison of the lowest order parts gives

$$G_i^0(v, D\psi^{\gamma_v}(p_v)\varsigma) = G_i^0(v, \varsigma). \tag{4.1}$$

Since the eigenvalues of $D\psi^{\gamma_v}(p_v)$ are all distinct, we can choose the coordinates (u, v, ξ, η) so that

$$D\psi^{\gamma_v}(p_v) = \operatorname{diag}(\lambda_1(v), \ldots, \lambda_m(v), \lambda_1^{-1}(v), \ldots, \lambda_m^{-1}(v)) \tag{4.2}$$

(see the proof of [[5], Proposition 6.1]). We recall that the fixed points p_v are non-resonant for any v belonging to a residual subset of V_h. Therefore from the identity (4.1) with (4.2), one can easily see that $G_i^0(v, \varsigma)$ are polynomials of m variables $\omega_j = \xi_j \eta_j$ with coefficients holomorphic in v. Consequently condition [C.2] implies that for any $v \in V_h$ fixed the following holds :

$$\det \left(\frac{\partial G_i^0}{\partial \omega_j} \right) \not\equiv 0. \tag{4.3}$$

Theorem 2 follows from Theorem 5.6 of [5]. Actually, Theorem 5.6 of [5] implies the existence of a system of holomorphic symplectic coordinates (u, v, ξ, η) in which the assertion (i) holds. Also the theorem shows that a function of v and ς is in parametrized normal form if it is invariant under Ψ^{γ_h}. Since any integral of $X_{H_1}, \dots, X_{H_k}$ is invariant under Ψ^{γ_h}, it is also in parametrized normal form. Hence the assertion (ii) is proved. To prove Theorem 5.6 of [5], we involved a typical small divisor problem when proving the convergence of a formal change of symplectic coordinates. But we overcame the difficulty by using the existence of additional integrals $G_1, \dots, G_m$ (see [4, 5]). Condition (4.3) plays an essential role in the proof of the theorem and also in the proof of other assertions in Theorem 2.

Finally we will prove Proposition 3.

Assume that $G_i(v, \varsigma)$ are holomorphic in $V_h \times \Xi$, V_h and Ξ being a neighbourhood of $v = h \in \mathbb{C}^k$ and that of $\varsigma = 0 \in \mathbb{C}^{2m}$ respectively. Condition [C.3] implies that

$$\operatorname{rank}\left(\frac{\partial(G_1, \dots, G_m)}{\partial(\varsigma_1, \dots, \varsigma_{2m})}\right) = m$$

on an open and dense subset of $V_h \times \Xi$. This does not necessarily imply the same condition with G_i replaced by their lowest order parts G_i^0. However we can prove the following

Lemma 1. ([5, Lemma 5.8].) *Let $G_i(v, \varsigma)$ $(i = 1, \dots, m)$ be convergent power series at $\varsigma = 0$ whose coefficients are holomorphic functions of v in a neighbourhood V_h of $v = h$. Assume that $v_1, \dots, v_k$ and $G_1, \dots, G_m$ are functionally independent. Then there exist $m(= n - k)$ functions $\widehat{G}_1, \dots, \widehat{G}_m$ which are polynomials of $G_1, \dots, G_m$ with coefficients holomorphic in v such that $\widehat{G}_i^0(v, \varsigma)$ satisfy the condition*

$$\operatorname{rank}\left(\frac{\partial(\widehat{G}_1^0, \dots, \widehat{G}_m^0)}{\partial(\varsigma_1, \dots, \varsigma_{2m})}\right) = m \tag{4.4}$$

on an open and dense subset of $V_h \times \mathbb{C}^{2m}$.

In the above, we take the neighbourhood V_h in the assumption to be sufficiently small. Otherwise the V_h in (4.4) has to be replaced by a neighbourhood smaller than V_h.

The condition (4.4) implies the existence of an open and dense subset $\widehat{V}_h$ of

V_h such that $\widehat{G}_i^0(v,\varsigma)$ are functionally independent functions of ς for any $v \in \widehat{V}_h$ fixed. Namely condition [C.4] follows. Moreover by their construction, we can write $\widehat{G}_i = P_i(G_1, \dots, G_m)$, P_i being polynomials of the form

$$P_i(z_1, \dots, z_m) = \sum_{\alpha} c_{i\,\alpha}(v) z^{\alpha} \tag{4.5}$$

by using multi-index notation. Here $c_{i\,\alpha}(v)$ are holomorphic functions of $v \in V_h$. If we replace v in (4.5) by H, then $\widehat{G}_i = P_i(G_1, \dots, G_m)$ is defined in a neighbourhood of $\{\Gamma_v\}_{V_h}$. Consequently holomorphic functions $H_1, \dots, H_k$ and $\widehat{G}_1, \dots, \widehat{G}_m$ satisfy condition [C.1] also. This completes the proof of Proposition 3. □

5. Proof of Theorem 3

In what follows, we will abbreviate the 'Birkhoff normal form' as 'normal form'.

First we will prove the assertion (i). In a neighbourhood of p_h, let us take a system of symplectic coordinates (u, v, ξ, η) established in Theorem 2. Our purpose is to show that the linear map $\varsigma' = f_0(\varsigma)$ satisfies (3.7) in this coordinate system. Notice that $G_i(v, \cdot)$ are invariant under any mapping $f \in \rho(\pi_1(\Gamma_v, p_v))$. It implies that the lowest order parts $G_i^0 := G_i^0(v, \cdot)$ are invariant under the linear map $f_0 = Df(p_v)$. Namely we have the identity

$$G_i^0 \circ f_0 = G_i^0. \tag{5.1}$$

From the relation (5.1), we can conclude the assertion (i) by using the argument by Ziglin [12] in the following way: First we set $\xi_j = \sqrt{\omega_j} e^{i\theta_j}$, $\eta_j = \sqrt{\omega_j} e^{-i\theta_j}$. Then $G_i^0 = G_i^0(v, \cdot)$ being in normal form, it follows from (5.1) that

$$\sum_{j=1}^{m} \frac{\partial (G_i^0 \circ f_0)}{\partial \omega'_j} \frac{\partial \omega'_j}{\partial \theta_\ell} = \frac{\partial G_i^0}{\partial \theta_\ell} = 0 \qquad (\ell = 1, \dots, m),$$

where $\omega'_j = (\xi_j \eta_j) \circ f_0(\varsigma)$. Since it follows from (4.3) that $\det(\partial G_i^0 / \partial \omega_j) \neq 0$ for any point ς in an open and dense subset of the neighbourhood of the origin, we can conclude that $\partial \omega'_j / \partial \theta_\ell \equiv 0$, which implies the functional dependence of ω'_j and $\omega_1, \dots, \omega_m$. Therefore ω'_i can be written as

$$\omega'_i = \sum_{j=1}^{m} \alpha_{ij} \omega_j \quad (\alpha_{ij} \in \mathbf{C}).$$

We note that the equation $\omega_i' = 0$ defines a pair of planes. Introducing new variables $q_i = \xi_i + \eta_i$ and $p_i = \xi_i - \eta_i$, we have

$$\omega_i' = \sum_{j=1}^{m} \frac{\alpha_{ij}}{4}(q_j^2 - p_j^2).$$

In order that the equation $\omega_i' = 0$ defines a pair of planes, the right-hand side has to be equal to $(\alpha_{ij}/4)(q_j^2 - p_j^2)$ for some $j \in \{1, \dots, m\}$. This implies that

$$\omega_i' = \alpha_{ij}\omega_j. \tag{5.2}$$

Moreover from the symplectic character of the linear map f_0, it follows that $\alpha_{ij} = 1$ or -1 in (5.2) and hence we have the relations (3.7). This completes the proof of (i).

Next we will prove the assertion (iii). By Theorem 2, the functions $G_i = G_i(v, \cdot)$ are in normal form. The condition [C.2] implies that they satisfy condition (4.3) for any v fixed in a sufficiently small neighbourhood of h. Let $f \in \rho(\pi_1(\Gamma_v, p_v))$ and $G_i^{(0)} = G_i^{(0)}(v, \cdot)$ functions defined by

$$G_i^{(0)} := G_i \circ f_0 \qquad (i = 1, \dots, m). \tag{5.3}$$

Then $G_i^{(0)}$ are also in normal form because f_0 satisfies (3.7). To prove that the 'nonlinear part' $\hat{f}$ is in normal form, we note the following lemma.

Lemma 2. *Let $f \in \mathrm{Symp}(\mathbb{C}^{2m}, 0)$ be of the form*

$$f(\varsigma) = \varsigma + O(|\varsigma|^{d+1})$$

with a positive integer d. Then it can be written as

$$f(\varsigma) = \varsigma + J\nabla W(\varsigma) + O(|\varsigma|^{2d+1}) = \exp X_W \circ (\varsigma + O(|\varsigma|^{2d+1}))$$

with

$$W(\varsigma) = W^{d+2} + \cdots + W^{2d+1},$$

where W^j are homogeneous polynomials in ς of degree j.

In the above, the notation $O(|\varsigma|^d)$ with a positive integer d denotes a (vector) function of ς whose components are convergent power series in ς containing only terms of order $\geq d$.

For the proof of Lemma 2, we refer to [[4], Lemma 2.7]. By this lemma, the transformation $\hat{f}$ can be expressed (at least formally) as

$$\hat{f} = \lim_{\nu\to\infty} f^{(\nu)}; \quad f^{(\nu)} = f_1 \circ f_2 \circ \cdots \circ f_\nu$$

with

$$f_\nu(\varsigma) = \exp X_W\,; \quad W = W^{d+2} + \cdots + W^{2d+1} \quad (d = 2^{\nu-1}). \tag{5.4}$$

Our purpose is to prove that each transformation f_ν is in normal form. To this end, let us first introduce, in addition to (5.3), the notation

$$G_i^{(\nu)} = G_i^{(0)} \circ f^{(\nu)}.$$

By the definition of $f^{(\nu)}$, $\hat{f}$ can be written as

$$\hat{f}(\varsigma) = f^{(\nu)} \circ (\varsigma + O(|\varsigma|^{d+1})) \qquad (d = 2^\nu\,;\ \nu = 0, 1, \dots),$$

where we assume $f^{(0)} = $ identity. From this relation, we have

$$G_i \circ f = G_i^{(0)} \circ \hat{f} = G_i^{(\nu)} + O(|\varsigma|^{s_i+d}) \qquad (d = 2^\nu), \tag{5.5}$$

where s_i are the degrees of the lowest order parts G_i^0 (as homogeneous polynomials in ς). Suppose that $f^{(\nu)}$ is in normal form. Recalling that $G_i^{(0)}$ are in normal form, $G_i^{(\nu)}$ are also in normal form. Since f_ν can be written as (5.4), we have

$$\begin{aligned} G_i^{(\nu+1)}(\varsigma) &= G_i^{(\nu)} \circ f_{\nu+1}(\varsigma) \\ &= G_i^{(\nu)}(\varsigma) + \{G_i^{(\nu)}(\varsigma), W(\varsigma)\} + O(|\varsigma|^{s_i+2d}) \qquad (d = 2^\nu). \end{aligned}$$

Let us define an operator P_N acting on the space of all power series by

$$P_N G := \sum_\alpha c_{\alpha\alpha}(v)\xi^\alpha \eta^\alpha \qquad \text{for} \quad G = \sum_{\alpha,\beta} c_{\alpha\beta}(v)\xi^\alpha\eta^\beta,$$

where we used the multi-index notation. Since $G_i = G_i \circ f$ are in normal form, it follows from (5.5) with ν replaced by $\nu+1$ that $G_i^{(\nu+1)}$ are in normal form up to terms of order $s_i + 2d - 1$ with $d = 2^\nu$, i.e., $G_i^{(\nu+1)} = P_N G_i^{(\nu+1)} + O(|\varsigma|^{s_i+2d})$. Since $P_N G_i^{(\nu)} = G_i^{(\nu)}$ and $P_N\{G_i^{(\nu)}, W\} = 0$, this implies that

$$\{G_i^{(\nu)}(\varsigma), W(\varsigma)\} = O(|\varsigma|^{s_i+2d}).$$

We set $\tilde{G}_i := G_i^{(\nu)}$ for convenience of notation. Then, by comparing the homogeneous parts of degree $s_i + \ell$ ($\ell = d, \dots, 2d-1$) in the equations above, each homogeneous polynomial $W^{\ell+2}$ satisfies the system of m equations

$$\sum_{j=1}^m \frac{\partial \tilde{G}_i^0}{\partial \omega_j} D_j W^{\ell+2} = F_i^\ell\,; \quad F_i^\ell := -\sum_{\mu=1}^{\ell-d} \{\tilde{G}_i^\mu, W^{\ell+2-\mu}\} \quad (i = 1, \dots, m),$$

where $D_j W^{\ell+2} := \{\omega_j, W^{\ell+2}\}$ and $\tilde{G}_i^{\mu}$ $(\mu = 0, 1, \ldots)$ are homogeneous parts of degree $s_i + \mu$ in the power series expansion of $\tilde{G}_i$ in ς. We note that $F_i^d = 0$ in the above. By the relation (5.1), the lowest order parts of $\tilde{G}_i = G_i^{(\nu)}$ and G_i are the same, i.e., $\tilde{G}_i^0 = G_i^0$. Therefore it follows from condition (4.3) that

$$D_j W^{d+2} \equiv 0 \qquad (j = 1, \ldots, m).$$

This implies that W^{d+2} is in normal form. Hence we can see that $F_i^{d+1} \equiv 0$ and can prove inductively that $W^{d+3}, \ldots, W^{2d+1}$ are in normal form. Therefore the transformation $f_{\nu+1}$ is in normal form and hence $f^{(\nu+1)} = f^{(\nu)} \circ f_{\nu+1}$ is also in normal form. It can be written as

$$f^{(\nu+1)}(\varsigma) = \exp X_W \,; \quad W = \sum_{\ell=3}^{2^{\nu+1}+1} W^{\ell}(\varsigma),$$

where W^{ℓ} are homogeneous polynomials in ς of degree ℓ and they are in normal form. Therefore, by induction it turns out that the transformation $\hat{f}$ is in normal form up to terms of arbitrary order, that is, $\hat{f}$ is in normal form. This completes the proof of (iii).

The assertion (ii) follows from (iii). In fact, from the assertion (i) it turns out that for $G = G(v, \cdot)$ the function $G' := G \circ f_0$ is in normal form and furthermore $G \circ f = G' \circ \hat{f} = G'$ because of the assertion (iii). Since G is an integral of $f \in \rho(\pi_1(\Gamma_v, p_v))$, we conclude that $G = G'$. Hence it turns out that G is invariant under the transformation f_0. This completes the proof of the assertion (ii). □

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Hidekazu Ito
Mathematical Institute
Tôhoku University
Sendai, 980, Japan